

GABOR REPRESENTATIONS OF EVOLUTION OPERATORS

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ABSTRACT. We perform a time-frequency analysis of Fourier multipliers and, more generally, pseudodifferential operators with symbols of Gevrey, analytic and ultra-analytic type. As an application we show that Gabor frames, which provide optimally sparse decompositions for Schrödinger-type propagators [14], surprisingly reveal to be an equally efficient tool for representing solutions to hyperbolic and parabolic-type differential equations with constant coefficients. In fact, the Gabor matrix representation of the corresponding propagator displays super-exponential decay away from the diagonal.

1. INTRODUCTION

A Gabor system is obtained by fixing a function $g \in L^2(\mathbb{R}^d)$ and considering the time-frequency-shifts

$$(1) \quad g_{m,n} = e^{2\pi i n x} g(x - m), \quad (m, n) \in \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d,$$

with $\alpha > 0$, $\beta > 0$. A Gabor system is a Gabor frame if there exist $A, B > 0$ such that

$$(2) \quad A \|f\|_{L^2}^2 \leq \sum_{m,n} |\langle f, g_{m,n} \rangle|^2 \leq B \|f\|_{L^2}^2$$

for every $f \in L^2(\mathbb{R}^d)$, see for example [18, 28, 38]. With respect to frames of wavelets, curvelets and shearlets, in (1) dilations are then replaced by modulations. From the analytic point of view, this provides symmetric properties with respect to the Fourier transform and a simple treatment, cf. [21].

Gabor frames turned out to be appropriate tools for many problems in time-frequency analysis, with relevant applications to signal processing and related issues in Numerical Analysis, see for example [9], [34], and references there. More recently, attention has been addressed to the analysis of the partial differential equations [2, 3, 4, 10, 11, 12, 13, 14, 16, 39, 40], let us also mention the pioneering work [17].

Aim of this paper is to represent in terms of Gabor frames the solutions of the Cauchy problems for a class of evolution operators with constant coefficients, including hyperbolic and parabolic operators. We shall prove in particular a super-exponential decay away from the diagonal, for the corresponding Gabor matrix.

To give a general setting for our results, let us denote by T the linear operator providing the solutions of a well-posed problem for a partial differential equation, in suitable function spaces. Generically, we expect that T is written in the form of

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pseudodifferential operator (PSDO), or Fourier integral operator (FIO). Via its action on a fixed frame, in particular a Gabor frame, T can be seen as an infinite matrix, with entries

$$(3) \quad T_{m',n',m,n} = \langle Tg_{m,n}, g_{m',n'} \rangle.$$

To deduce results of sparsity for this representation, one is then led to estimate the decay properties of $T_{m',n',m,n}$, for large values of the indices m', n', m, n .

In this order of ideas [5, 6] obtained optimal sparsity for strictly hyperbolic problems in the frame of the curvelets. The operator T was a classical FIO with phase function homogeneous of degree 1 in the dual variables, cf. [27, Vol. 4],

By parallel arguments the authors of [14] proved optimal sparsity for Schrödinger equations in the Gabor frames. In this case, the operator T was a FIO with phase function of quadratic type, cf. [1], having natural action on the Gabor atoms $g_{m,n}$.

The treatment of hyperbolic problems in Gabor frames remained outside the results of [14]. In this regards, the example of the transport equation with non-constant coefficients, standard test for numerical schemes, is somewhat discouraging. In fact, for a fixed time $t > 0$, the operator T solving the Cauchy problem turns out to be a change of variables, and Gabor frames are not suited to follow the corresponding non-linear propagation of singularities.

Because of this, our attack to hyperbolic equation via Gabor frames will be limited to operators with constant coefficients. Nevertheless, with respect to [5, 6], our results will be stronger in the following aspects:

- we shall treat hyperbolic equations, not necessarily strictly hyperbolic, of any order and in an arbitrary number of space variables;
- our class of equation will include also parabolic equations;
- the decay of the Gabor matrix will be of super-exponential type.

As a simple example, let us consider the Cauchy problem for the wave equation:

$$(4) \quad \begin{aligned} \partial_t^2 u - \Delta_x u &= 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) &= u_0(x), & \partial_t u(0, x) = u_1(x). \end{aligned}$$

We may express the solution in the form

$$(5) \quad u(t, x) = T_t u_1(x) + \partial_t T_t u_0(x),$$

where T_t is the Fourier multiplier

$$(6) \quad T_t f(x) = \int e^{2\pi i x \xi} \sigma_t(\xi) \widehat{f}(\xi) d\xi,$$

with symbol

$$(7) \quad \sigma_t(\xi) = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}, \quad \xi \in \mathbb{R}^d.$$

Taking the Gaussian as window function g and discretizing consequently $T = T_t$, we get a Gabor matrix satisfying

$$(8) \quad |T_{m',n',m,n}| \leq C e^{-\epsilon(|m-m'|^2 + |n-n'|^2)}.$$

We address to the sequel of the paper for the dependence of C and ϵ on $t > 0$. With respect to the approach by FIOs, the ξ -dependent phase functions are here absorbed into the symbol (7). Because of its oscillations, $\sigma_t(\xi)$ then belongs to the non-standard Hörmander's class $S_{0,0}^0$, defined in general by the estimates

$$(9) \quad |\partial^\alpha \sigma(z)| \leq C_\alpha, \quad z = (x, \xi) \in \mathbb{R}^{2d}.$$

As well known, PSDOs with symbol in $S_{0,0}^0$ have unpredictable action on singular supports. In spite of this, the estimates are very strong.

Let us describe in short how (8) can be obtained in the general context of symbols depending on $z = (x, \xi)$. The proof will be largely based on [22, 24, 32], where the authors show that for every PSDO with symbol $\sigma \in S_{0,0}^0$, taking as window any function g in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, we have

$$(10) \quad |T_{m',n',m,n}| \leq C_N (1 + |m - m'|^2 + |n - n'|^2)^{-N}$$

for N as large as we want. Here, inspired by the properties of the symbol in (7) we introduce an analytic version of the class $S_{0,0}^0$, by assuming for $s \geq 0$

$$(11) \quad |\partial^\alpha \sigma(z)| \leq C^{|\alpha|+1} (\alpha!)^s, \quad z \in \mathbb{R}^{2d},$$

for a constant $C > 0$ depending on σ . We say that the symbol σ is Gevrey when $s > 1$, analytic when $s = 1$, and ultra-analytic if $s < 1$. As we will show, such estimates are satisfied by the symbols of the fundamental solutions for a large class of evolution operators with constant coefficients, with $0 \leq s < 1$. In particular, we have $s = 0$ for the wave equation (4), and $s = 1 - 1/(2k)$, $k \in \mathbb{N} \setminus \{0\}$, for the generalized heat equation:

$$(12) \quad \partial_t u + (-\Delta_x)^k u = 0, \quad u(0, x) = u_0.$$

Let T be a PSDO with symbol σ satisfying (11) for some $s \geq 0$, and define $r = \min\{2, 1/s\}$. Fix as a window the Gaussian, or more generally any function g belonging to the Gelfand-Shilov classes $S_r^r(\mathbb{R}^d)$. Then we shall prove that

$$(13) \quad |T_{m',n',m,n}| \leq C e^{-\epsilon(|m - m'| + |n - n'|)^r}.$$

Accordingly, every column or row of the Gabor matrix, rearranged in decreasing order, displays a similar decay, i.e. we obtain an *exponential-type sparsity*.

In the case of the Gevrey-analytic symbols, i.e. $s \geq 1$ in (11) and $r \leq 1$ in (13), the estimates follow from the results in [24]. In view of the applications, our attention will be mainly addressed to the ultra-analytic case, i.e. $0 \leq s < 1$ in (11), $r > 1$ in (13); in particular from (13) with $r = 2$ we have (8) for the wave equation.

The plan of the paper is the following. In Section 2 we provide the necessary preliminaries, concerning Gelfand-Shilov spaces, time-frequency representations and modulation spaces. Section 3 is devoted to the study of the symbols in (11). The corresponding PSDOs are considered in Section 4, where we prove (13) and give boundedness results in modulation and Gelfand-Shilov spaces. Applications to evolution equations are given in Section 5.

2. PRELIMINARIES

2.1. Notations. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$.

Euclidean norm of $x \in \mathbb{R}^d$ is given by $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. We write $xy = x \cdot y$ for the scalar product on \mathbb{R}^d , for $x, y \in \mathbb{R}^d$.

For multiindices $\alpha, \beta \in \mathbb{N}^d$, we have $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\alpha! = \alpha_1! \cdots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $x \in \mathbb{R}^d$, and, for $\beta \leq \alpha$, i.e. $\beta_j \leq \alpha_j$, $j \in \{1, 2, \dots, d\}$, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}$. The letter C denotes a positive constant, not necessarily the same at every appearance.

The operator of partial differentiation ∂ is given by

$$\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$$

for all multiindices $\alpha \in \mathbb{N}^d$ and all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. If f and g are smooth enough, then the Leibniz formula holds

$$\partial^\alpha(fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f(x) \partial^\beta g(x).$$

Observe

$$(14) \quad \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|},$$

and

$$(15) \quad |\alpha|! \leq d^{|\alpha|} \alpha!.$$

The Fourier transform is normalized to be $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t) e^{-2\pi i t \xi} dt$.

Translation and modulation operators, T and M are defined by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_x f(\cdot) = e^{2\pi i x \cdot} f(\cdot), \quad x \in \mathbb{R}^d.$$

For $z = (x, \xi)$ we shall also write

$$\pi(z)f = M_\xi T_x f.$$

The following relations hold

$$M_y T_x = e^{2\pi i x y} T_x M_y, \quad (T_x f)^\wedge = M_{-x} \hat{f}, \quad (M_x f)^\wedge = T_x \hat{f}, \quad x, y \in \mathbb{R}^d, f, g \in L^2(\mathbb{R}^d).$$

For $0 < p \leq \infty$ and a weight m , the space $\ell_m^p(\Lambda)$ is the space of sequences $a = \{a_\lambda\}_{\lambda \in \Lambda}$ on a lattice Λ , such that

$$\|a\|_{\ell_m^p} := \left(\sum_{\lambda \in \Lambda} |a_\lambda|^p m(\lambda)^p \right)^{1/p} < \infty$$

(with obvious changes when $p = \infty$).

Throughout the paper, we shall use the notation $A \lesssim B$ to express the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

2.2. Gelfand-Shilov Spaces. The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ does not give enough information about how fast a function $f \in \mathcal{S}(\mathbb{R}^d)$ and its derivatives decay at infinity. This is the main motivation to use subspaces of the Schwartz class, so-called Gelfand-Shilov type spaces, introduced in [20] and now available also in a textbook [30]. Let us recall their definition and main properties.

Definition 2.1. Let there be given $s, r \geq 0$. A function $f \in \mathcal{S}(\mathbb{R}^d)$ is in the Gelfand-Shilov type space $S_r^s(\mathbb{R}^d)$ if there exist constants $A, B > 0$ such that

$$(16) \quad |x^\alpha \partial^\beta f(x)| \lesssim A^{|\alpha|} B^{|\beta|} (\alpha!)^r (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d.$$

The space $S_r^s(\mathbb{R}^d)$ is nontrivial if and only if $r + s > 1$, or $r + s = 1$ and $r, s > 0$ [20]. So the smallest nontrivial space with $r = s$ is provided by $S_{1/2}^{1/2}(\mathbb{R}^d)$. Every function of the type $P(x)e^{-a|x|^2}$, with $a > 0$ and $P(x)$ polynomial on \mathbb{R}^d , is in the class $S_{1/2}^{1/2}(\mathbb{R}^d)$. We observe the trivial inclusions $S_{r_1}^{s_1}(\mathbb{R}^d) \subset S_{r_2}^{s_2}(\mathbb{R}^d)$ for $s_1 \leq s_2$ and $r_1 \leq r_2$. Moreover, if $f \in S_r^s(\mathbb{R}^d)$, also $x^\delta \partial^\gamma f$ belongs to the same space for every fixed δ, γ .

The action of the Fourier transform on $S_r^s(\mathbb{R}^d)$ interchanges the indices s and r , as explained in the following theorem.

Theorem 2.2. For $f \in \mathcal{S}(\mathbb{R}^d)$ we have $f \in S_r^s(\mathbb{R}^d)$ if and only if $\hat{f} \in S_s^r(\mathbb{R}^d)$.

Therefore for $s = r$ the spaces $S_s^s(\mathbb{R}^d)$ are invariant under the action of the Fourier transform.

Theorem 2.3. Assume $s > 0, r > 0, s+r \geq 1$. For $f \in \mathcal{S}(\mathbb{R}^d)$, the following conditions are equivalent:

a) $f \in S_r^s(\mathbb{R}^d)$.

b) There exist constants $A, B > 0$, such that

$$\|x^\alpha f\|_{L^\infty} \lesssim A^{|\alpha|} (\alpha!)^r \quad \text{and} \quad \|\xi^\beta \hat{f}\|_{L^\infty} \lesssim B^{|\beta|} (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d.$$

c) There exist constants $A, B > 0$, such that

$$\|x^\alpha f\|_{L^\infty} \lesssim A^{|\alpha|} (\alpha!)^r \quad \text{and} \quad \|\partial^\beta f\|_{L^\infty} \lesssim B^{|\beta|} (\beta!)^s, \quad \alpha, \beta \in \mathbb{N}^d.$$

d) There exist constants $h, k > 0$, such that

$$\|f e^{h|x|^{1/r}}\|_{L^\infty} < \infty \quad \text{and} \quad \|\hat{f} e^{k|\xi|^{1/s}}\|_{L^\infty} < \infty.$$

A suitable window class for weighted modulation spaces (see the subsequent Definition 2.6) is the Gelfand-Shilov type space $\Sigma_1^1(\mathbb{R}^d)$, consisting of functions $f \in \mathcal{S}(\mathbb{R}^d)$ such that for every constant $A > 0$ and $B > 0$

$$(17) \quad |x^\alpha \partial^\beta f(x)| \lesssim A^{|\alpha|} B^{|\beta|} \alpha! \beta!, \quad \alpha, \beta \in \mathbb{N}^d.$$

We have $S_s^s(\mathbb{R}^d) \subset \Sigma_1^1(\mathbb{R}^d) \subset S_1^1(\mathbb{R}^d)$ for every $s < 1$. Observe that the characterization of Theorem 2.3 can be adapted to $\Sigma_1^1(\mathbb{R}^d)$ by replacing the words “there exist” by “for every” and taking $r = s = 1$.

Let us underline the following property, which exhibits two equivalent ways of expressing the decay of a continuous function f on \mathbb{R}^d . This follows immediately from [30, Proposition 6.1.5], where the property was formulated for $f \in \mathcal{S}(\mathbb{R}^d)$. For sake of

clarify, we shall detail the proof showing the mutual dependence between the constants ϵ and C below.

Proposition 2.4 ([30, Proposition 6.1.5]). *Consider $r > 0$ and let h be a continuous function on \mathbb{R}^d . Then the following conditions are equivalent:*

(i) *There exists a constant $\epsilon > 0$ such that*

$$(18) \quad |h(x)| \lesssim e^{-\epsilon|x|^{\frac{1}{r}}}, \quad x \in \mathbb{R}^d,$$

(ii) *There exists a constant $C > 0$ such that*

$$(19) \quad |x^\alpha h(x)| \lesssim C^{|\alpha|}(\alpha!)^r, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

Proof. We re-write (18) in the form

$$(20) \quad |h(x)|^{\frac{1}{r}} \lesssim e^{-\frac{\epsilon}{r}|x|^{\frac{1}{r}}}, \quad x \in \mathbb{R}^d.$$

In turn, (20) can be re-written as

$$(21) \quad \sup_{x \in \mathbb{R}^d} \sum_{n=0}^{\infty} \left(\frac{\epsilon}{r}\right)^n (n!)^{-1} |x|^{\frac{n}{r}} |h(x)|^{\frac{1}{r}} < \infty.$$

Hence the sequence of the terms of the series is uniformly bounded, as well as the sequence of the r -th powers:

$$\frac{\epsilon^{rn}}{r^{rn}} (n!)^{-r} |x|^n |h(x)|, \quad n \in \mathbb{N},$$

and we obtain

$$|x|^n |h(x)| \lesssim \frac{\epsilon^{rn}}{r^{rn}} (n!)^r, \quad x \in \mathbb{R}^d, n = 0, 1, \dots$$

Hence, writing $|\alpha| = n$ and applying (15):

$$|x^\alpha h(x)| \lesssim \left(\frac{rd}{\epsilon}\right)^{r|\alpha|} (\alpha!)^r = C^{|\alpha|}(\alpha!)^r, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d,$$

where $C = \left(\frac{rd}{\epsilon}\right)^r$. Therefore (19) is proved.

In the opposite direction, let (19) be satisfied. Using the following inequalities

$$|x|^n \leq \sum_{|\alpha|=n} \frac{n!}{\alpha!} |x^\alpha|, \quad \sum_{|\alpha|=n} \frac{n!}{\alpha!} = d^n, \quad \text{and} \quad \alpha! \leq |\alpha|!,$$

and the assumption (19) we obtain

$$|x|^n |h(x)| \leq \sum_{|\alpha|=n} \frac{n!}{\alpha!} |x^\alpha h(x)| \lesssim \sum_{|\alpha|=n} \frac{n!}{\alpha!} C^{|\alpha|}(\alpha!)^r \leq C^n (n!)^r d^n = (dC)^n (n!)^r,$$

for every $x \in \mathbb{R}^d$, $n \in \mathbb{N}$. Therefore the sequence

$$(dC)^{-n} (n!)^{-r} |x|^n |h(x)|, \quad n \in \mathbb{N}$$

is uniformly bounded for $x \in \mathbb{R}^d$, as well as the sequence

$$(dC)^{-\frac{n}{r}} (n!)^{-1} |x|^{\frac{n}{r}} |h(x)|^{\frac{1}{r}}, \quad n \in \mathbb{N}.$$

If we choose $\epsilon = q(dC)^{-\frac{1}{r}}$, for a fixed $q \in (0, 1)$, we conclude

$$e^{\epsilon|x|^{\frac{1}{r}}} |h(x)|^{\frac{1}{r}} = \sum_{n=0}^{\infty} q^n (dC)^{-\frac{n}{r}} (n!)^{-1} |x|^{\frac{n}{r}} |h(x)|^{\frac{1}{r}} \lesssim \sum_{n=0}^{\infty} q^n.$$

This is (20), hence (18) and the proof is complete. \square

It follows from this proof the precise relation between the constants ϵ and C . Indeed, assuming (18), then (19) is satisfied with $C = \left(\frac{rd}{\epsilon}\right)^r$. Viceversa, (19) implies (18) for any $\epsilon < r(dC)^{-\frac{1}{r}}$. The bound is sharp for $d = 1$. Also, it follows from the proof that the constant implicit in the notation \lesssim in (18) depends only on the corresponding one in (19) and viceversa.

The strong dual spaces of $S_r^s(\mathbb{R}^d)$ and $\Sigma_1^1(\mathbb{R}^d)$ are called spaces of tempered ultra-distributions and denoted by $(S_r^s)'(\mathbb{R}^d)$ and $(\Sigma_1^1)'(\mathbb{R}^d)$, respectively. Notice that they contain the space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$.

The spaces $S_r^s(\mathbb{R}^d)$ are nuclear spaces [29]. This property yields a kernel theorem for Gelfand-Shilov spaces, cf. [37].

Theorem 2.5. *There exists an isomorphism between the space of linear continuous maps T from $S_r^s(\mathbb{R}^d)$ to $(S_r^s)'(\mathbb{R}^d)$ and $(S_r^s)'(\mathbb{R}^{2d})$, which associates to every T a kernel $K_T \in (S_r^s)'(\mathbb{R}^{2d})$ such that*

$$\langle Tu, v \rangle = \langle K_T, v \otimes \bar{u} \rangle, \quad \forall u, v \in S_r^s(\mathbb{R}^d).$$

K_T is called the kernel of T .

2.3. Time-frequency representations. We recall the basic concepts of time-frequency analysis and refer the reader to [21] for the full details. Consider a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (the so-called *window*). The short-time Fourier transform (STFT) of f with respect to g is $V_g f(z) = \langle f, \pi(z)g \rangle$, $z = (x, \xi) \in \mathbb{R}^{2d}$. The short-time Fourier transform is well-defined whenever the bracket $\langle \cdot, \cdot \rangle$ makes sense for dual pairs of function or (ultra-)distribution spaces, in particular for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, $f, g \in L^2(\mathbb{R}^d)$, $f \in (\Sigma_1^1)'(\mathbb{R}^d)$ and $g \in \Sigma_1^1(\mathbb{R}^d)$ or $f \in (S_r^s)'(\mathbb{R}^d)$ and $g \in S_r^s(\mathbb{R}^d)$. Let us recall the covariance formula for the short-time Fourier transform that will be used in the sequel

$$(22) \quad V_g(M_\eta T_y f)(x, \xi) = e^{-2\pi i(\xi-\eta)y} (V_g f)(x-y, \xi-\eta), \quad x, y, \xi, \eta \in \mathbb{R}^d.$$

Another time-frequency representation we shall use is the (cross-)Wigner distribution of $f, g \in L^2(\mathbb{R}^d)$, defined as

$$(23) \quad W(f, g)(x, \xi) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t \xi} dt.$$

If we set $\check{g}(t) = g(-t)$, then the relation between cross-Wigner distribution and short-time Fourier transform is provided by

$$(24) \quad W(f, g)(x, \xi) = 2^d e^{4\pi i x \xi} V_{\check{g}} f(2x, 2\xi).$$

For the discrete description of function spaces and operators we use Gabor frames. Let $\Lambda = A\mathbb{Z}^{2d}$ with $A \in GL(2d, \mathbb{R})$ be a lattice of the time-frequency plane. The set

of time-frequency shifts $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ for a non-zero $g \in L^2(\mathbb{R}^d)$ is called a Gabor system. The set $\mathcal{G}(g, \Lambda)$ is a Gabor frame, if there exist constants $A, B > 0$ such that

$$(25) \quad A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d).$$

If (25) is satisfied, then there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$, such that $\mathcal{G}(\gamma, \Lambda)$ is a frame, and every $f \in L^2(\mathbb{R}^d)$ possesses the frame expansions

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. In conclusion, we list some results about time-frequency analysis of Gelfand-Shilov functions, cf. [8, 15, 25, 35]:

$$(26) \quad f, g \in S_s^s(\mathbb{R}^d), s \geq 1/2 \Rightarrow V_g f \in S_s^s(\mathbb{R}^{2d}),$$

$$(27) \quad f, g \in S_s^s(\mathbb{R}^d), s \geq 1/2 \Rightarrow W(f, g) \in S_s^s(\mathbb{R}^{2d}),$$

$$(28) \quad f, g \in \Sigma_1^1(\mathbb{R}^d) \Rightarrow W(f, g) \in \Sigma_1^1(\mathbb{R}^{2d}).$$

If $g \in S_s^s(\mathbb{R}^d)$, $s \geq 1/2$, then

$$(29) \quad f \in S_s^s(\mathbb{R}^d) \Leftrightarrow |V_g(f)(z)| \lesssim e^{-\epsilon|z|^{1/s}} \text{ for some } \epsilon > 0.$$

2.4. Modulation Spaces. Weighted modulation spaces measure the decay of the STFT on the time-frequency (phase space) plane and were introduced by Feichtinger in the 80's [19], for weight of sub-exponential growth at infinity. The study of weights of exponential growth at infinity was developed in [15, 35], so that we can now recall their definition for weights with at most exponential decay at infinity.

Weight Functions. In the sequel v will always be a continuous, positive, even, submultiplicative function (submultiplicative weight), i.e., $v(0) = 1$, $v(z) = v(-z)$, and $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z, z_1, z_2 \in \mathbb{R}^{2d}$. Submultiplicativity implies that $v(z)$ is *dominated* by an exponential function, i.e.

$$(30) \quad \exists C, k > 0 \text{ such that } 1 \leq v(z) \leq Ce^{k|z|}, \quad z \in \mathbb{R}^{2d}.$$

For example, every weight of the form

$$(31) \quad v(z) = e^{s|z|^b} (1 + |z|)^a \log^r(e + |z|)$$

for parameters $a, r, s \geq 0$, $0 \leq b \leq 1$ satisfies the above conditions.

We denote by $\mathcal{M}_v(\mathbb{R}^d)$ the space of v -moderate weights on \mathbb{R}^d ; these are measurable positive functions m satisfying $m(z + \zeta) \leq Cv(z)m(\zeta)$ for every $z, \zeta \in \mathbb{R}^d$.

Definition 2.6. Given $g \in \Sigma_1^1(\mathbb{R}^d)$, a weight function $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, and $1 \leq p, q \leq \infty$, the modulation space $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered ultra-distributions $f \in (\Sigma_1^1)'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}(\mathbb{R}^d)$ is

$$(32) \quad \|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \xi)|^p m(x, \xi)^p dx \right)^{q/p} d\xi \right)^{1/q}$$

(obvious changes if $p = \infty$ or $q = \infty$).

We observe that for $f, g \in \Sigma_1^1(\mathbb{R}^d)$ the above integral is convergent and thus $\Sigma_1^1(\mathbb{R}^d) \subset M_m^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, cf. [15], with dense inclusion when $p, q < \infty$, cf. [8]. When $p = q$, we simply write $M_m^p(\mathbb{R}^d)$ instead of $M_m^{p,p}(\mathbb{R}^d)$. The spaces $M_m^{p,q}(\mathbb{R}^d)$ are Banach spaces and every nonzero $g \in M_v^1(\mathbb{R}^d)$ yields an equivalent norm in (32) and so $M_m^{p,q}(\mathbb{R}^d)$ is independent on the choice of $g \in M_v^1(\mathbb{R}^d)$.

We recall the inversion formula for the STFT (see ([21, Proposition 11.3.2] and [15, Proposition 2.6] for exponential weights): assume $g \in M_v^1(\mathbb{R}^d) \setminus \{0\}$, $f \in M_m^{p,q}(\mathbb{R}^d)$, then

$$(33) \quad f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} V_g f(x, \xi) M_\xi T_x g \, dx \, d\xi,$$

and the equality holds in $M_m^{p,q}(\mathbb{R}^d)$ (observe that $M_v^1(\mathbb{R}^d) \subset M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, so $\|g\|_2 < \infty$).

3. GEVREY-ANALYTIC AND ULTRA-ANALYTIC SYMBOLS

In this section we characterize the smoothness and the growth of a function f on \mathbb{R}^d in terms of the decay of its STFT $V_g f$, for a suitable window g . In the proofs we shall detail the relations among the constants $C_f, C_g, C_{f,g}$ and ϵ which come into play, since it could be useful for constructing efficient algorithms related to these objects.

Theorem 3.1. *Consider $s > 0$, $m \in \mathcal{M}_v(\mathbb{R}^d)$, $g \in M_{v \otimes 1}^1(\mathbb{R}^d) \setminus \{0\}$ such that there exists $C_g > 0$,*

$$(34) \quad \|\partial^\alpha g\|_{L_v^1(\mathbb{R}^d)} \lesssim C_g^{|\alpha|} (\alpha!)^s, \quad \alpha \in \mathbb{N}^d.$$

For $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ the following conditions are equivalent:

(i) There exists a constant $C_f > 0$ such that

$$(35) \quad |\partial^\alpha f(x)| \lesssim m(x) C_f^{|\alpha|} (\alpha!)^s, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

(ii) There exists a constant $C_{f,g} > 0$ such that

$$(36) \quad |\xi^\alpha V_g f(x, \xi)| \lesssim m(x) C_{f,g}^{|\alpha|} (\alpha!)^s, \quad (x, \xi) \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^d.$$

(iii) There exists a constant $\epsilon > 0$ such that

$$(37) \quad |V_g f(x, \xi)| \lesssim m(x) e^{-\epsilon |\xi|^{\frac{1}{s}}}, \quad (x, \xi) \in \mathbb{R}^{2d}, \alpha \in \mathbb{N}^d.$$

If the equivalent conditions (35), (36), (37) are satisfied, we say that f is a Gevrey symbol when $s > 1$, an analytic symbol if $s = 1$ and an ultra-analytic symbol when $s < 1$.

Proof. (i) \Rightarrow (ii). Since $f, T_x g \in \mathcal{C}^\infty(\mathbb{R}^d)$, for every $x \in \mathbb{R}^d$, we can use Leibniz formula and write

$$\partial^\alpha (f T_x g)(t) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f(t) T_x (\partial^\beta g)(t), \quad t \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

Let us estimate the $L^1(\mathbb{R}^d)$ -norm of the products $\partial^{\alpha-\beta} f T_x(\partial^\beta g)$. Using the positivity and v -moderateness of the weight m , Hölder inequality and the assumptions (35) and (34),

$$\begin{aligned} \|\partial^{\alpha-\beta} f T_x(\partial^\beta g)\|_{L^1} &\leq \|\partial^{\alpha-\beta} f\|_{L_{1/m}^\infty} \|m T_x(\partial^\beta g)\|_{L^1} \\ &\lesssim C_f^{|\alpha-\beta|} ((\alpha-\beta)!)^s \|v(\cdot-x) \partial^\beta g(\cdot-x)\|_{L^1} m(x) \\ &\lesssim C_f^{|\alpha-\beta|} ((\alpha-\beta)!)^s C_g^{|\beta|} (\beta!)^s m(x) \\ &\leq C^{|\alpha|} (\alpha!)^s m(x) \end{aligned}$$

where $C := \max\{C_f, C_g\}$ and we used $(\alpha-\beta)!\beta! \leq \alpha!$. These estimates tell us, in particular, that $V_g f(x, \xi) = \mathcal{F}(f T_x g)(\xi)$ is well-defined. We can exchange partial derivatives and Fourier transform as follows

$$\xi^\alpha V_g f(x, \xi) = \frac{1}{(2\pi i)^{|\alpha|}} \mathcal{F}(\partial^\alpha(f T_x g))(\xi) = \frac{1}{(2\pi i)^{|\alpha|}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathcal{F}(\partial^{\alpha-\beta} f T_x(\partial^\beta g))(\xi).$$

Using

$$|\mathcal{F}(\partial^{\alpha-\beta} f T_x(\partial^\beta g))(\xi)| \leq \|\mathcal{F}(\partial^{\alpha-\beta} f T_x(\partial^\beta g))\|_{L^\infty} \leq \|\partial^{\alpha-\beta} f T_x(\partial^\beta g)\|_{L^1},$$

the majorizations above and (14),

$$|\xi^\alpha V_g f(x, \xi)| \lesssim \frac{m(x)}{(2\pi)^{|\alpha|}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C^{|\alpha|} (\alpha!)^s = \frac{m(x)}{\pi^{|\alpha|}} C^{|\alpha|} (\alpha!)^s.$$

This proves (36), with constant $C_{f,g} := \frac{\max\{C_f, C_g\}}{\pi}$.

(ii) \Rightarrow (i). We use the inversion formula (33), observing that (36) implies $f \in M_{m \otimes 1}^{\infty, 1}(\mathbb{R}^d)$, so that the equality in (33) holds a.e. and we can assume that it holds everywhere since f is smooth. Let us consider the partial derivatives of f and exchange them with the integrals, the estimates below will provide the justification of this operation. So, formally, we can write

$$\partial^\alpha f(t) = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} V_g f(x, \xi) \partial^\alpha (M_\xi T_x g)(t) dx d\xi, \quad t \in \mathbb{R}^d.$$

Using Leibniz formula $\partial^\alpha (M_\xi T_x g) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi i \xi)^\beta M_\xi T_x(\partial^{\alpha-\beta} g)$ we estimate

$$|\partial^\alpha f(t)| \leq \frac{1}{\|g\|_2^2} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi)^{|\beta|} \int_{\mathbb{R}^{2d}} |V_g f(x, \xi) \xi^\beta| \cdot |T_x \partial^{\alpha-\beta} g(t)| dx d\xi, \quad t \in \mathbb{R}^d.$$

We set

$$I_{\alpha, \beta}(t) := \int_{\mathbb{R}^{2d}} |V_g f(x, \xi) \xi^\beta| \cdot |T_x \partial^{\alpha-\beta} g(t)| dx d\xi$$

and prove that, for every fixed $t \in \mathbb{R}^d$, $I_{\alpha, \beta}(t)$ are absolutely convergent integrals. Using $1 + |\xi|^{d+1} \leq c_d \sum_{|\gamma| \leq d+1} |\xi^\gamma|$, where c_d depends only on the dimension d , and

assumption (36),

$$\begin{aligned} I_{\alpha,\beta}(t) &= \int_{\mathbb{R}^{2d}} |V_g f(x, \xi)| \xi^\beta \left| \frac{1 + |\xi|^{d+1}}{1 + |\xi|^{d+1}} T_x \partial^{\alpha-\beta} g(t) \right| dx d\xi \\ &\lesssim \int_{\mathbb{R}^{2d}} \sum_{|\gamma| \leq d+1} |V_g f(x, \xi)| \xi^{\beta+\gamma} \left| \frac{1}{1 + |\xi|^{d+1}} T_x \partial^{\alpha-\beta} g(t) \right| dx d\xi \\ &\lesssim \sum_{|\gamma| \leq d+1} C_{f,g}^{|\beta+\gamma|} ((\beta+\gamma)!)^s \left(\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^{d+1}} d\xi \right) \left(\int_{\mathbb{R}^d} m(x) |T_x \partial^{\alpha-\beta} g(t)| dx \right). \end{aligned}$$

Now, $(\beta+\gamma)! \leq 2^{|\beta|+|\gamma|} (\beta!) (\gamma!) \leq 2^{d+1} (d+1)! 2^{|\beta|} \beta!$ and $C_{f,g}^{|\beta+\gamma|} = C_{f,g}^{d+1} C_{f,g}^{|\beta|}$ so, forgetting about the constants depending only on d , f and g ,

$$\begin{aligned} I_{\alpha,\beta}(t) &\lesssim C_{f,g}^{|\beta|} 2^{s\beta} (\beta!)^s \left(\int_{\mathbb{R}^d} m(x) |T_x \partial^{\alpha-\beta} g(t)| dx \right) \\ &\lesssim C_{f,g}^{|\beta|} 2^{s\beta} (\beta!)^s m(t) \left(\int_{\mathbb{R}^d} v(x-t) |\partial^{\alpha-\beta} g(t-x)| dx \right) \\ &= C_{f,g}^{|\beta|} 2^{s\beta} (\beta!)^s m(t) \|\partial^{\alpha-\beta} g\|_{L_v^1}. \end{aligned}$$

The estimate above and the assumption (34) on g allow the following majorization:

$$\begin{aligned} |\partial^\alpha f(t)| &\lesssim \frac{m(t)}{\|g\|_2^2} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi)^{|\beta|} (2^s C_{f,g})^{|\beta|} (\beta!)^s C_g^{|\alpha-\beta|} ((\alpha-\beta)!)^s \\ &\leq \frac{m(t)}{\|g\|_2^2} (\alpha!)^s \max\{C_{f,g}, C_g\}^{|\alpha|} 2^{s|\alpha|} (2\pi)^{|\alpha|} 2^{|\alpha|} \\ &\lesssim m(t) C_f^\alpha (\alpha!)^s, \end{aligned}$$

where $C_f := 2^{s+2} \pi \max\{C_{f,g}, C_g\}$, and we used $(\beta)! (\alpha-\beta)! \leq \alpha!$ and (14).

(ii) \Leftrightarrow (iii). The equivalence is an immediate consequence of Proposition 2.4 and the subsequent remarks on the constants, where, for every fixed $x \in \mathbb{R}^d$, we choose $h(\xi) := V_g f(x, \xi) / m(x)$ and $r = s$. \square

A natural question is whether we may find window functions satisfying (34). To this end, we recall the following characterization of Gelfand-Shilov spaces.

Proposition 3.2. *Let $g \in \mathcal{S}(\mathbb{R}^d)$. We have $g \in S_r^s(\mathbb{R}^d)$, with $s, r > 0$, $r+s \geq 1$, if and only if there exist constants $A > 0$, $\epsilon > 0$ such that*

$$|\partial^\alpha g(x)| \lesssim A^{|\alpha|} (\alpha!)^s e^{-\epsilon|x|^\frac{1}{r}}, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

We have $g \in \Sigma_1^1(\mathbb{R}^d)$ if and only if, for every $A > 0$, $\epsilon > 0$,

$$|\partial^\alpha g(x)| \lesssim A^{|\alpha|} \alpha! e^{-\epsilon|x|}, \quad x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d.$$

Proof. The first part of the statement is in [30, Proposition 6.1.7]. For the second part, the assumption $g \in \Sigma_1^1(\mathbb{R}^d)$ means that for every $A > 0$, $B > 0$,

$$|x^\beta \partial^\alpha g(x)| \lesssim A^{|\alpha|} B^{|\beta|} \alpha! \beta!, \quad x \in \mathbb{R}^d, \alpha, \beta \in \mathbb{N}^d.$$

Therefore the function $h = \frac{\partial^\alpha g}{A^{|\alpha|}\alpha!}$ satisfies (19) in Proposition 2.4 for every $C > 0$, and the estimate (18) is then satisfied for every $\epsilon > 0$ (see the observations after the proof of Proposition 2.4 for the uniformity of the constants). This gives the claim. \square

Hence every $g \in S_r^s(\mathbb{R}^d)$ with $s > 0$, $0 < r < 1$, $s + r \geq 1$, satisfies (34) for every submultiplicative weight v (see (30)). The same holds true if $g \in \Sigma_1^1(\mathbb{R}^d)$ and $s \geq 1$.

4. ALMOST DIAGONALIZATION FOR PSEUDODIFFERENTIAL OPERATORS

In this section we extend some results of [24] to the case of pseudodifferential operators having (ultra-)analytic symbols. Our result covers also the Gevrey-analytic case which was already discussed in [24].

The Weyl form σ^w of a pseudodifferential operator (so-called Weyl operator or Weyl transform) with symbol $\sigma(x, \xi)$ on \mathbb{R}^{2d} is formally defined by

$$(38) \quad \sigma^w f(x) = \int_{\mathbb{R}^d} \sigma\left(\frac{x+y}{2}, \eta\right) e^{2\pi i(x-y)\eta} f(y) dy d\eta.$$

Using the Wigner distribution (23), we can recast the definition as follows

$$(39) \quad \langle \sigma^w f, g \rangle = \langle \sigma, W(f, g) \rangle.$$

Since for $f, g \in \mathcal{S}(\mathbb{R}^d)$ we have $W(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$, then (39) defines for $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ a continuous map $\sigma^w : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. The Schwartz kernel K of σ^w is given by

$$(40) \quad K(x, y) = \mathcal{F}_{x \rightarrow x-y}^{-1} \sigma\left(\frac{x+y}{2}, \xi\right) \in \mathcal{S}'(\mathbb{R}^{2d}).$$

On the other hand, in view of the kernel theorem in $\mathcal{S} - \mathcal{S}'$, every linear continuous map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^{2d})$ can be represented by means of a kernel $K \in \mathcal{S}'(\mathbb{R}^{2d})$, hence by means of (38), (39) with symbol

$$(41) \quad \sigma(x, \xi) = \mathcal{F}_{y \rightarrow \xi} K\left(x + \frac{y}{2}, x - \frac{y}{2}\right).$$

Same arguments are valid in Gelfand-Shilov classes with $s = r \geq 1/2$ (see [29, 37]). Namely, considering $\sigma \in (S_s^s)'(\mathbb{R}^{2d})$, $f, g \in S_s^s(\mathbb{R}^d)$ in (38), (39), we have $\sigma^w : S_s^s(\mathbb{R}^d) \rightarrow (S_s^s)'(\mathbb{R}^d)$ continuously, in view of (27). The kernel K of σ^w , given by (40), belongs to $(S_s^s)'(\mathbb{R}^{2d})$. In the opposite direction, in view of Theorem 2.5, every linear continuous map from $S_s^s(\mathbb{R}^d)$ to $(S_s^s)'(\mathbb{R}^d)$ can be represented in the form (38), (39), with $\sigma \in (S_s^s)'(\mathbb{R}^{2d})$ given by (41). The same holds for the couple of spaces $\Sigma_1^1, (\Sigma_1^1)'$.

The crucial relation between the action of the Weyl operator σ^w on time-frequency shifts and the short-time Fourier transform of its symbol, contained in [22, Lemma 3.1] can now be extended to Gelfand-Shilov spaces and their dual spaces as follows.

Lemma 4.1. *Consider $s \geq 1/2$, $g \in S_s^s(\mathbb{R}^d)$, $\Phi = W(g, g)$. Then, for $\sigma \in (S_s^s)'(\mathbb{R}^{2d})$,*

$$(42) \quad |\langle \sigma^w \pi(z)g, \pi(w)g \rangle| = \left| V_\Phi \sigma\left(\frac{z+w}{2}, j(w-z)\right) \right| = |V_\Phi \sigma(u, v)|$$

and

$$(43) \quad |V_\Phi \sigma(u, v)| = \left| \langle \sigma^w \pi\left(u - \frac{1}{2}j^{-1}(v)\right) g, \pi\left(u + \frac{1}{2}j^{-1}(v)\right) g \rangle \right|,$$

where $j(z_1, z_2) = (z_2, -z_1)$. Moreover, the same results hold true if we replace the space $S_s^s(\mathbb{R}^d)$ with the space $\Sigma_1^1(\mathbb{R}^d)$.

Proof. Since $\Phi = W(g, g) \in S_s^s(\mathbb{R}^{2d})$ for $g \in S_s^s(\mathbb{R}^d)$ the duality $\langle \sigma, \pi(u, v) \Phi \rangle_{(S_s^s)' \times S_s^s}$ is well-defined so that the short-time Fourier transform $V_\Phi \sigma(u, v)$ makes sense. The same pattern applies to the case $g \in \Sigma_1^1(\mathbb{R}^d)$. The rest of the proof is analogous to [22, Lemma 3.1]. \square

We now exhibit a characterization of smooth symbols $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ for σ^w in terms of the continuous Gabor matrix. The symbol σ belongs to the classes defined in Section 3, the dimension being now $2d$.

Theorem 4.2. *Let $s \geq 1/2$, and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. If $1/2 \leq s < 1$ consider a window function $g \in S_s^s(\mathbb{R}^d)$; otherwise, if $s \geq 1$, assume either $g \in \Sigma_1^1(\mathbb{R}^d)$, or $g \in S_s^s(\mathbb{R}^d)$ and the following growth condition on the weight v :*

$$v(z) \lesssim e^{\epsilon|z|^{1/s}}, \quad z \in \mathbb{R}^{2d},$$

for every $\epsilon > 0$. Then the following properties are equivalent for $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d})$:

(i) The symbol σ satisfies

$$(44) \quad |\partial^\alpha \sigma(z)| \lesssim m(z) C^{|\alpha|} (\alpha!)^s, \quad \forall z \in \mathbb{R}^{2d}, \forall \alpha \in \mathbb{N}^{2d}.$$

(ii) There exists $\epsilon > 0$ such that

$$(45) \quad |\langle \sigma^w \pi(z) g, \pi(w) g \rangle| \lesssim m\left(\frac{w+z}{2}\right) e^{-\epsilon|w-z|^{1/s}}, \quad \forall z, w \in \mathbb{R}^{2d}.$$

Proof. (i) \Rightarrow (ii). Proposition 3.2 applied to the window Φ in Lemma 4.1 and the assumptions on v (recall also (30)) imply that Φ satisfies the assumptions of Theorem 3.1. Hence, using the equivalence (35) \Leftrightarrow (37), the assumption (44) is equivalent to the following decay estimate of the corresponding short-time Fourier transform

$$|V_\Phi \sigma(u, v)| \lesssim m(u) e^{-\epsilon|v|^{1/s}}, \quad u, v \in \mathbb{R}^{2d},$$

for a suitable $\epsilon > 0$, which combined with (42) yields

$$|V_\Phi \sigma\left(\frac{z+w}{2}, j(w-z)\right)| \lesssim m\left(\frac{w+z}{2}\right) e^{-\epsilon|j(w-z)|^{1/s}} = m\left(\frac{w+z}{2}\right) e^{-\epsilon|w-z|^{1/s}}$$

that is (ii).

(ii) \Rightarrow (i). Relation (43) and the decay assumption (45) give

$$\begin{aligned} |V_\Phi \sigma(u, v)| &= \left| \langle \sigma^w \pi\left(u - \frac{1}{2} j^{-1}(v)\right) g, \pi\left(u + \frac{1}{2} j^{-1}(v)\right) g \rangle \right| \\ &\lesssim m(u) e^{-\epsilon|j^{-1}(v)|^{1/s}} = m(u) e^{-\epsilon|v|^{1/s}} \end{aligned}$$

and using the equivalence (35) \Leftrightarrow (37) we obtain the claim. \square

Of course, from (45) we deduce the discrete Gabor matrix decay in (48) below. The viceversa requires more effort. Indeed, we appeal to a recent result obtained by Gröchenig and Lyubarskii in [23]. They find sufficient conditions on the lattice $\Lambda = A\mathbb{Z}^2$, $A \in GL(2, \mathbb{R})$, such that $g = \sum_{k=0}^n c_k H_k$, with H_k Hermite function, forms a

so-called Gabor (super)frame $\mathcal{G}(g, \Lambda)$. Besides they prove the existence of dual windows γ that belong to the space $S_{1/2}^{1/2}(\mathbb{R})$ (cf. [23, Lemma 4.4]). This theory transfers to the d -dimensional case by taking a tensor product $g = g_1 \otimes \cdots \otimes g_d \in S_{1/2}^{1/2}(\mathbb{R}^d)$ of windows as above, which defines a Gabor frame on the lattice $\Lambda_1 \times \cdots \times \Lambda_d$ and possesses a dual window $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_d$ in the same space $\in S_{1/2}^{1/2}(\mathbb{R}^d)$. Let us simply call *Gabor super-frame* $\mathcal{G}(g, \Lambda)$ for $L^2(\mathbb{R}^d)$ a Gabor frame with the above properties. The Gabor super-frames are the key for the discretization of the kernel in (45). First, we need the preliminary result below, which reflects the algebra property of Gelfand-Shilov spaces. For $s \geq 1/2$, $\epsilon > 0$, we define the following weight functions:

$$(46) \quad w_{s,\epsilon}(z) := e^{-\epsilon|z|^{\frac{1}{s}}}, \quad z \in \mathbb{R}^{2d}.$$

Lemma 4.3. *Let $\Lambda \subset \mathbb{R}^{2d}$ be a lattice of \mathbb{R}^{2d} . Then the sampling $\{w_{s,\epsilon}(\lambda)\}_{\lambda \in \Lambda}$ of (46) (defined on \mathbb{R}^{2d}) satisfies*

$$(47) \quad (w_{s,\epsilon} * w_{s,\epsilon})(\lambda) := \sum_{\nu \in \Lambda} w_{s,\epsilon}(\lambda - \nu) w_{s,\epsilon}(\nu) \lesssim \begin{cases} w_{s,\epsilon}(\lambda), & \text{for } s \geq 1 \\ w_{s,\epsilon 2^{-1/s}}(\lambda), & \text{for } \frac{1}{2} \leq s < 1. \end{cases}$$

Proof. For $s \geq 1$ the result immediately follows by the submultiplicativity of $w_{s,\epsilon}$:

$$(w_{s,\epsilon} * w_{s,\epsilon})(\lambda) = \sum_{\nu \in \Lambda} w_{s,\epsilon}(\lambda - \nu) w_{s,\epsilon}(\nu) \leq w_{s,\epsilon}(\lambda) \sum_{\nu \in \Lambda} w_{s,\epsilon}(-\nu) w_{s,\epsilon}(\nu) \lesssim w_{s,\epsilon}(\lambda).$$

For $1/2 \leq s < 1$, we use the arguments of [21, Lemma 11.1.1(c)]. For $\lambda \in \Lambda$, we divide the lattice Λ into the subsets $N_\lambda = \{\nu \in \Lambda : |\lambda - \nu| \leq |\lambda|/2\}$ and $N_\lambda^c = \{\nu \in \Lambda : |\lambda - \nu| > |\lambda|/2\}$. For $\nu \in N_\lambda$, $|\nu| \geq |\lambda|/2$ and $|\nu|^{1/s} \geq (|\lambda|/2)^{1/s}$, so

$$(w_{s,\epsilon} * w_{s,\epsilon})(\lambda) \leq e^{-(\epsilon 2^{-\frac{1}{s}})|\lambda|^{\frac{1}{s}}} \left(\sum_{\nu \in N_\lambda} e^{-\epsilon|\lambda-\nu|^{\frac{1}{s}}} + \sum_{\nu \in N_\lambda^c} e^{-\epsilon|\nu|^{\frac{1}{s}}} \right) \lesssim e^{-(\epsilon 2^{-\frac{1}{s}})|\lambda|^{\frac{1}{s}}}.$$

This concludes the proof. \square

Theorem 4.4. *Let $\mathcal{G}(g, \Lambda)$ a Gabor super-frame for $L^2(\mathbb{R}^d)$. Consider $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $s \geq 1/2$, and a symbol $\sigma \in \mathcal{C}^\infty(\mathbb{R}^{2d})$. Then the following properties are equivalent:*

- (i) *There exists $\epsilon > 0$ such that the estimate (45) holds.*
- (ii) *There exists $\epsilon > 0$ such that*

$$(48) \quad |\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle| \lesssim m \left(\frac{\lambda + \mu}{2} \right) e^{-\epsilon|\lambda - \mu|^{\frac{1}{s}}}, \quad \forall \lambda, \mu \in \Lambda.$$

Proof. It remains to show that (ii) \Rightarrow (i). The pattern of [22, Theorem 3.2] can be adapted to this proof by using a Gabor super-frame $\mathcal{G}(g, \Lambda)$, with a dual window $\gamma \in S_{1/2}^{1/2}(\mathbb{R}^d)$.

Let \mathcal{Q} be a symmetric relatively compact fundamental domain of the lattice $\Lambda \subset \mathbb{R}^{2d}$. Given $w, z \in \mathbb{R}^{2d}$, we can write them uniquely as $w = \lambda + u$, $z = \mu + u'$, for $\lambda, \mu \in \Lambda$

and $u, u' \in \mathcal{Q}$. Using the Gabor reproducing formula for the time-frequency shift $\pi(u)g \in S_{1/2}^{1/2}(\mathbb{R}^d)$ we can write

$$\pi(u)g = \sum_{\nu \in \Lambda} \langle \pi(u)g, \pi(\nu)\gamma \rangle \pi(\nu)g.$$

Inserting the prior expansions in the assumption (48),

$$\begin{aligned} & |\langle \sigma^w \pi(\mu + u')g, \pi(\lambda + u)g \rangle| \\ & \leq \sum_{\nu, \nu' \in \Lambda} |\langle \sigma^w \pi(\mu + \nu')g, \pi(\lambda + \nu)g \rangle| |\langle \pi(u')g, \pi(\nu')\gamma \rangle| |\langle \pi(u)g, \pi(\nu)\gamma \rangle| \\ (49) \quad & \lesssim \sum_{\nu, \nu' \in \Lambda} m\left(\frac{\lambda + \mu + \nu + \nu'}{2}\right) e^{-\epsilon|\lambda + \nu - \mu - \nu'|^{\frac{1}{s}}} |V_\gamma g(\nu' - u')| |V_\gamma g(\nu - u)|. \end{aligned}$$

Since the window functions g, γ are both in $S_{1/2}^{1/2}(\mathbb{R}^d)$, the STFT $V_\gamma g$ is in $S_{1/2}^{1/2}(\mathbb{R}^{2d})$ in view of (26). Thus there exists $h > 0$ such that $|V_\gamma g(z)| \lesssim e^{-h|z|^2}$, for every $z \in \mathbb{R}^{2d}$. In particular, being \mathcal{Q} relatively compact and $u \in \mathcal{Q}$,

$$|V_\gamma g(\nu - u)| \lesssim e^{-h|\nu - u|^2} \leq \sup_{u \in \mathcal{Q}} e^{-h|\nu - u|^2} \lesssim e^{-h|\nu|^2}.$$

The assumption $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ yields

$$m\left(\frac{\lambda + \mu + \nu + \nu'}{2}\right) \lesssim m\left(\frac{\lambda + \mu}{2}\right) v\left(\frac{\nu}{2}\right) v\left(\frac{\nu'}{2}\right)$$

and, for every $0 < \tilde{h} < h$,

$$v\left(\frac{\nu}{2}\right) e^{-h|\nu|^2} \lesssim e^{-\tilde{h}|\nu|^2}, \quad \forall \nu \in \Lambda.$$

Inserting these estimates in (49),

$$\begin{aligned} & |\langle \sigma^w \pi(\mu + u')g, \pi(\lambda + u)g \rangle| \\ & \lesssim m\left(\frac{\lambda + \mu}{2}\right) \sum_{\nu, \nu' \in \Lambda} e^{-\epsilon|\lambda + \nu - \mu - \nu'|^{\frac{1}{s}}} v\left(\frac{\nu}{2}\right) e^{-h|\nu|^2} v\left(\frac{\nu'}{2}\right) e^{-h|\nu'|^2} \\ & \lesssim m\left(\frac{\lambda + \mu}{2}\right) \sum_{\nu, \nu' \in \Lambda} e^{-\epsilon|\lambda + \nu - \mu - \nu'|^{\frac{1}{s}}} e^{-\tilde{h}|\nu|^2} e^{-\tilde{h}|\nu'|^2} \\ (50) \quad & \leq m\left(\frac{\lambda + \mu}{2}\right) \sum_{\nu, \nu' \in \Lambda} e^{-b(|\lambda + \nu - \mu - \nu'|^{\frac{1}{s}} + |\nu|^{\frac{1}{s}} + |\nu'|^{\frac{1}{s}})} \end{aligned}$$

with $b = \epsilon$ for $s > 1/2$ whereas $b = \min\{\epsilon, \tilde{h}\}$ for $s = 1/2$ (there may be a loss of decay). We observe that the raw (50) can be rewritten as

$$m\left(\frac{\lambda + \mu}{2}\right) (w_{s,b} * w_{s,b} * w_{s,b})(\lambda - \mu),$$

where $w_{s,b}(\lambda) = e^{-b|\lambda|^{1/s}}$. Now we apply Lemma 4.3 twice and we obtain

$$(51) \quad (50) \lesssim m \left(\frac{\lambda + \mu}{2} \right) e^{-\tilde{\epsilon}|\lambda - \mu|^{\frac{1}{s}}}$$

with $\tilde{\epsilon} = b$ when $s \geq 1$ whereas $\tilde{\epsilon} = b2^{-2/s}$ when $1/2 \leq s < 1$. Again, observe the loss of decay for $1/2 \leq s < 1$.

If $w, z \in \mathbb{R}^{2d}$ and $w = \lambda + u, z = \mu + u', \lambda, \mu \in \Lambda, u, u' \in \mathcal{Q}$, then $\lambda - \mu = w - z + u' - u$ and $u' - u \in \mathcal{Q} - \mathcal{Q}$, which is a relatively compact set, thus

$$(52) \quad e^{-\tilde{\epsilon}|\lambda - \mu|^{\frac{1}{s}}} \lesssim \sup_{u \in \mathcal{Q} - \mathcal{Q}} e^{-\tilde{\epsilon}|w - z + u|^{\frac{1}{s}}} \lesssim e^{-\tilde{\epsilon}|w - z|^{\frac{1}{s}}}.$$

Finally, the v -moderateness of the weight $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, together with the fact that v is continuous and the set $\mathcal{Q} + \mathcal{Q}$ is relatively compact, let us write

$$(53) \quad \begin{aligned} m \left(\frac{\lambda + \nu}{2} \right) &= m \left(\frac{w + z}{2} - \frac{u + u'}{2} \right) \lesssim m \left(\frac{w + z}{2} \right) v \left(-\frac{u + u'}{2} \right) \\ &\lesssim m \left(\frac{w + z}{2} \right) \sup_{u \in \mathcal{Q} + \mathcal{Q}} v \left(\frac{u}{2} \right) \lesssim m \left(\frac{w + z}{2} \right) \end{aligned}$$

Combining the estimates (52) and (53) with (51) we obtain (45), with the parameter $\epsilon = \tilde{\epsilon}$ which appears in (51). \square

4.1. Sparsity of the Gabor matrix. The operators σ^w which satisfy Theorem 4.2, say with $m = v = 1$, enjoy a fundamental sparsity property. Indeed, let $\mathcal{G}(g, \Lambda)$ be a Gabor frame for $L^2(\mathbb{R}^d)$, with $g \in S_s^s(\mathbb{R}^d)$, $s \geq 1/2$. Then, as we saw,

$$(54) \quad |\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle| \leq Ce^{-\epsilon|\lambda - \mu|^{\frac{1}{s}}}, \quad \forall \lambda, \mu \in \Lambda,$$

with suitable constants $C > 0, \epsilon > 0$. This gives at once an exponential-type sparsity, in the sense precised by the following proposition (cf. [5, 26] for the more standard notion of super-polynomial sparsity).

Proposition 4.5. *Let the Gabor matrix $\langle \sigma^w \pi(\mu)g, \pi(\lambda)g \rangle$ satisfy (54). Then it is sparse in the following sense. Let a be any column or raw of the matrix, and let $|a|_n$ be the n -largest entry of the sequence a . Then, $|a|_n$ satisfies*

$$|a|_n \leq Ce^{-\epsilon n^{\frac{1}{2ds}}}, \quad n \in \mathbb{N}$$

for some constants $C > 0, \epsilon > 0$.

Proof. By a discrete analog of Proposition 2.4 it suffices to prove that

$$n^\alpha |a|_n \leq C^{\alpha+1}(\alpha!)^{2ds}, \quad \alpha \in \mathbb{N}.$$

On the other hand we have

$$n^{\frac{1}{p}} \cdot |a|_n \leq \|a\|_{\ell^p},$$

for every $0 < p \leq \infty$. Hence by (54) and setting $p = 1/\alpha$ we obtain

$$n^\alpha |a|_n \leq \left(\sum_{\lambda \in \Lambda} e^{-\epsilon p |\lambda - \mu|^{\frac{1}{s}}} \right)^{\frac{1}{p}} = \left(\sum_{\lambda \in \Lambda} e^{-\epsilon p |\lambda|^{\frac{1}{s}}} \right)^{\frac{1}{p}}.$$

Let \mathcal{Q} be a fundamental domain of the lattice Λ . Then if $x \in \lambda + \mathcal{Q}$, $\lambda \in \Lambda$, we have $|x| \leq |\lambda| + C_0$, therefore $|x|^{1/s} \leq C_1(|\lambda|^{1/s} + 1)$. Hence

$$\begin{aligned} \sum_{\lambda \in \Lambda} e^{-\epsilon p |\lambda|^{1/s}} &\leq C_2 \int_{\mathbb{R}^{2d}} e^{-\epsilon p |x|^{1/s}} dx = \int_{\mathbb{S}^{2d-1}} d\sigma \int_0^{+\infty} e^{-\epsilon p \rho^{1/s}} \rho^{2d-1} d\rho \\ &= \frac{C_3 s}{(\epsilon p)^{2ds}} \int_0^{+\infty} e^{-t} t^{2ds-1} dt = \frac{C_3 s \Gamma(2ds)}{(\epsilon p)^{2ds}} = \frac{C_4}{p^{2ds}} \end{aligned}$$

Finally, by Stirling's formula,

$$n^\alpha |a|_n \leq \frac{C_4^{1/p}}{p^{2ds}} \leq C_5^{\alpha+1} (\alpha!)^{2ds}.$$

□

4.2. Boundedness of pseudodifferential operators on modulation and Gelfand-Shilov spaces. Although not necessary for our numerical applications, we list some continuity results on modulation and Gelfand-Shilov spaces which follow easily from our basic Theorem 4.2.

Proposition 4.6. *Let $s \geq 1/2$ and consider a symbol $\sigma \in C^\infty(\mathbb{R}^{2d})$ satisfying the estimates*

$$(55) \quad |\partial^\alpha \sigma(z)| \lesssim C^{|\alpha|} (\alpha!)^s,$$

for some $C > 0$. Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and, if $s \geq 1$, assume the weight v satisfies

$$v(z) \lesssim e^{\epsilon |z|^{1/s}}, \quad z \in \mathbb{R}^{2d}$$

for every $\epsilon > 0$. Then the Weyl operator σ^w extends to a bounded operator on $\mathcal{M}_m^{p,q}(\mathbb{R}^d)$.

Proof. Let $g \in \Sigma_1^1(\mathbb{R}^d)$ with $\|g\|_{L^2} = 1$. From the inversion formula (33),

$$V_g(\sigma^w f)(u) = \int_{\mathbb{R}^{2d}} \langle \sigma^w \pi(z) g, \pi(u) g \rangle V_g f(z) dz.$$

The desired result thus follows if we can prove that the map $M(\sigma)$ defined by

$$M(\sigma)G(u) = \int_{\mathbb{R}^{2d}} \langle \sigma^w \pi(z) g, \pi(u) g \rangle G(z) dz$$

is continuous from $L_m^{p,q}(\mathbb{R}^{2d})$ into $L_m^{p,q}(\mathbb{R}^{2d})$. The characterization of Theorem 4.2 assures the existence of an $\epsilon > 0$, such that

$$|V_g(\sigma^w f)| = |M(\sigma)V_g(f)| \lesssim (w_{s,\epsilon} * |V_g f|)(u), \quad u \in \mathbb{R}^{2d}$$

where the weight function $w_{s,\epsilon}$ on \mathbb{R}^{2d} is defined in (46). The desired conclusion then follows from the relation $L_m^{p,q} * L_v^1 \hookrightarrow L_m^{p,q}$, for $w_{s,\epsilon} \in L_v^1$ by the growth assumption on v (if $s < 1$, (30) suffices).

□

Proposition 4.7. *Let $s \geq 1/2$, and consider a symbol $\sigma \in C^\infty(\mathbb{R}^{2d})$ that satisfies (55). Then the Weyl operator σ^w is bounded on $S_s^s(\mathbb{R}^d)$.*

Proof. Fix a window function $g \in S_{1/2}^{1/2}(\mathbb{R}^d)$. For $f \in S_s^s(\mathbb{R}^d)$ we have $V_g f \in S_s^s(\mathbb{R}^{2d})$ in view of (26). Hence there exists a constant $h > 0$ such that $|V_g f|(z) \lesssim e^{-h|z|^{1/s}}$, for every $z \in \mathbb{R}^{2d}$. Thus, taking $\tilde{\epsilon} = \min\{\epsilon, h\}$ and using a continuous version of Lemma 4.3, for every $u \in \mathbb{R}^{2d}$,

$$|V_g(\sigma^w f)(u)| \lesssim (w_{s,\epsilon} * w_{s,h})(u) \lesssim (w_{s,\tilde{\epsilon}} * w_{s,\tilde{\epsilon}})(u) \lesssim \begin{cases} w_{s,\tilde{\epsilon}}(u), & \text{for } s \geq 1 \\ w_{s,\tilde{\epsilon}2^{-1/s}}(u), & \text{for } \frac{1}{2} \leq s < 1. \end{cases}$$

Using (29) we obtain the claim. \square

Boundedness results for pseudodifferential operators on Gelfand-Shilov spaces are contained in [36].

5. APPLICATIONS TO EVOLUTION EQUATIONS

In this section we apply the almost diagonalization result to the propagators of Cauchy problems for constant coefficient hyperbolic and parabolic-type equations.

We consider an operator

$$(56) \quad P(\partial_t, D_x) = \partial_t^m + \sum_{k=1}^m a_k(D_x) \partial_t^{m-k}, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d$$

where $a_k(\xi)$, $1 \leq k \leq m$, are polynomials. They may be non-homogeneous, and their degree may be arbitrary. Here we set $D_{x_j} = \frac{1}{2\pi i} \partial_{x_j}$, $j = 1, \dots, d$.

We are interested in the forward Cauchy problem

$$\begin{cases} P(\partial_t, D_x)u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ \partial_t^k u(0, x) = u_k(x), & 0 \leq k \leq m-1, \end{cases}$$

where $u_k \in \mathcal{S}(\mathbb{R}^d)$, $0 \leq k \leq m-1$. Hence we suppose that the *forward Hadamard-Petrowsky condition* is satisfied: *There exists a constant $C > 0$ such that*

$$(57) \quad (\tau, \xi) \in \mathbb{C} \times \mathbb{R}^d, \quad P(i\tau, \xi) = 0 \implies \operatorname{Im} \tau \geq -C.$$

This is a sufficient and necessary condition for the above Cauchy problem with Schwartz data to be well posed [31, Section 3.10]. By taking the Fourier transform with respect to x and using classical results for the fundamental solution to ordinary differential operators [33, pp. 126-127], one sees that the solution is then given by

$$u(t, x) = \sum_{k=0}^{m-1} \partial_t^k E(t, \cdot) * \left(u_{m-1-k} + \sum_{j=1}^{m-k-1} a_j(D_x) u_{m-k-1-j} \right).$$

Here $E(t, x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \sigma(t, \xi)$, where $\sigma(t, \xi)$ is the unique solution to

$$\left(\partial_t^m + \sum_{k=1}^m a_k(\xi) \partial_t^{m-k} \right) \sigma(t, \xi) = \delta(t)$$

supported in $[0, +\infty) \times \mathbb{R}^d$. The distribution $E(t, x)$ is therefore the fundamental solution of P supported in $[0, +\infty) \times \mathbb{R}^d$.

The study of the Cauchy problem is therefore reduced to that of the Fourier multiplier

$$(58) \quad \sigma^w(t, D_x) = \sigma(t, D_x)f = \mathcal{F}^{-1}\sigma(t, \cdot)\mathcal{F}f = E(t, \cdot)*f.$$

(For Fourier multipliers the Weyl and Kohn-Nirenberg quantizations give the same operator).

Example 5.1. *Here are some classical operators and the symbols $\sigma(t, \xi)$ of the corresponding propagators for $t \geq 0$ ($\sigma(t, \xi) = 0$ for $t < 0$):*

Wave operator $\partial_t^2 - \Delta$; $\sigma(t, \xi) = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$.

Klein-Gordon operator $\partial_t^2 - \Delta + m^2$, with $m > 0$; $\sigma(t, \xi) = \frac{\sin(t\sqrt{4\pi^2|\xi|^2+m^2})}{\sqrt{4\pi^2|\xi|^2+m^2}}$.

Heat operator $\partial_t - \Delta$; $\sigma(t, \xi) = e^{-4\pi^2|\xi|^2t}$.

We now introduce an assumption under which the multiplier $\sigma(t, D_x)$ falls in the class of pseudodifferential operators considered above.

Assume that there are constants $C > 0$, $r \geq 1$ such that

$$(59) \quad (\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^d, \quad P(i\tau, \zeta) = 0 \implies \operatorname{Im} \tau \geq -C(1 + |\operatorname{Im} \zeta|)^r.$$

It is clear that this condition is stronger than the forward Hadamard-Petrowsky condition.

Theorem 5.2. *Assume P satisfies (59) for some $C > 0$, $r \geq 1$. Then the symbol $\sigma(t, \xi)$ of the corresponding propagator $\sigma(t, D_x)$ in (58) satisfies the following estimates:*

$$(60) \quad |\partial_\xi^\alpha \sigma(t, \xi)| \leq C^{(t+1)|\alpha|+t}(\alpha!)^\mu, \quad \xi \in \mathbb{R}^d, \quad t \geq 0, \quad \alpha \in \mathbb{N}^d,$$

with $\mu = 1 - 1/r$, for a new constant $C > 0$.

Proof. It is well known (see e.g. [7, Proposition 1.3.2, Lemma 1.3.3], [31, Section 3.10]) that $\sigma(t, \xi)$ extends to an entire analytic function in the second variable, and that the estimates (59) imply the bound

$$(61) \quad |\sigma(t, \zeta)| \leq e^{Ct(1+|\operatorname{Im} \zeta|)^r}, \quad \zeta \in \mathbb{C}^d.$$

Now, given $\xi \in \mathbb{R}^d$, we consider the polydisk $B(\xi, R) = \prod_{j=1}^d B_j(\xi_j, R) = \{\zeta \in \mathbb{C}^d : |\zeta_j - \xi_j| \leq R, 1 \leq j \leq d\}$, with $R = (1 + |\alpha|)^{1/r}$, $\alpha \in \mathbb{N}^d$. Observe that (61) implies

$$(62) \quad \sup_{\zeta \in B(\xi, R)} |\sigma(t, \zeta)| \leq e^{Ct(1+\sqrt{d}R)^r}.$$

The Cauchy's Generalized Integral Formula

$$\partial_\xi^\alpha \sigma(t, \xi) = \frac{\alpha!}{(2\pi i)^d} \int \cdots \int_{\partial B_1(\xi_1, R) \times \cdots \times \partial B_d(\xi_d, R)} \frac{\sigma(t, \zeta_1, \dots, \zeta_d)}{(\zeta_1 - \xi_1)^{\alpha_1+1} \cdots (\zeta_d - \xi_d)^{\alpha_d+1}} d\zeta_1 \cdots d\zeta_d$$

and the estimate (62) yield

$$(63) \quad |\partial_\xi^\alpha \sigma(t, \xi)| \leq \frac{\alpha! e^{Ct(1+\sqrt{d}R)^r}}{R^{|\alpha|}} \leq C_1^{t(|\alpha|+1)} \frac{\alpha!}{(1 + |\alpha|)^{|\alpha|/r}}$$

Using Stirling formula and $1/r = 1 - \mu$,

$$\frac{1}{(1 + |\alpha|)^{|\alpha|/r}} \leq \frac{C_2^{|\alpha|}}{(\alpha!)^{1-\mu}}$$

which combined with (63) provides the desired majorization (60). \square

Observe that the assumption $r \geq 1$ in the above theorem implies $0 \leq \mu < 1$.

Combining Theorem 5.2 and Theorem 4.2 we obtain at once our main application.

Theorem 5.3. *Assume P satisfies (59) for some $C > 0$, $r \geq 1$, and set $s = \max\{1/2, 1 - 1/r\}$. If $g \in S_s^s(\mathbb{R}^d)$ then $\sigma(t, D_x)$ in (58) satisfies*

$$(64) \quad |\langle \sigma(t, D_x) \pi(z)g, \pi(w)g \rangle| \leq C e^{-\epsilon|w-z|^{\frac{1}{s}}}, \quad \forall z, w \in \mathbb{R}^{2d},$$

for some $\epsilon > 0$ and for a new constant $C > 0$. The constants ϵ and C are uniform when t lies in bounded subsets of $[0, +\infty)$.

Notice that in the above theorem $1/2 \leq s < 1$, so that we always obtain super-exponential decay, i.e. $1/s > 1$ in (64).

We now show that, if $P(\partial_t, D_x)$ is any hyperbolic operator, then (59) is satisfied with $r = 1$, and hence the above theorem applies with Gaussian decay ($s = 1/2$ in (64)), for windows $g \in S_{1/2}^{1/2}(\mathbb{R}^d)$.

We recall that the operator $P(\partial_t, D_x)$ is called hyperbolic with respect to t if the direction $N = (1, 0, \dots, 0) \in \mathbb{R} \times \mathbb{R}^d$ is non characteristic for P , (i.e. its principal symbol – the higher order homogeneous part in the symbol – does not vanish at N) and P satisfies the forward Hadamard-Petrowsky condition (57). It follows then that the operators $a_j(D_x)$ in (56) must have degree $\leq j$ and P has order m .

The wave and Klein-Gordon operators are of course the most important examples of hyperbolic operators. We emphasize, however, that P is *not* required to be strictly hyperbolic, namely the roots of the principal symbol are allowed to coincide.

For example the operator $P = \partial_t^2 - \sum_{j,k=1}^d a_{j,k} \partial_{x_j} \partial_{x_k}$, is hyperbolic if the matrix $a_{j,k}$ is real, symmetric and positive *semi*-definite.

Proposition 5.4. *Assume $P(\partial_t, D_x)$ is hyperbolic with respect to t . Then the condition (59) is satisfied with $r = 1$ for some $C > 0$, and hence*

$$(65) \quad |\langle \sigma(t, D_x) \pi(z)g, \pi(w)g \rangle| \leq C e^{-\epsilon|w-z|^2}, \quad \forall z, w \in \mathbb{R}^{2d},$$

if $g \in S_{1/2}^{1/2}(\mathbb{R}^d)$, for some $\epsilon > 0$ and for a new constant $C > 0$.

Proof. Denote by P_m the principal symbol of P and $\Gamma(P, N)$ the component of N in $\{\theta \in \mathbb{R} \times \mathbb{R}^d : P_m(\theta) \neq 0\}$. It follows from the hyperbolicity assumption (see e.g. [27, Proposition 12.4.4]) that the symbol $Q(\tau, \xi) = P(i\tau, \xi)$ of $P(\partial_t, D_x)$ satisfies

$$Q(\Xi + \tau N + \sigma \theta) \neq 0 \quad \text{if } \Xi \in \mathbb{R}^{d+1}, \text{ Im } \tau < -C, \text{ Im } \sigma \leq 0, \theta \in \Gamma(P, N),$$

for the same constant C which appears in (57). We then deduce that

$$(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^d, \quad Q(\tau, \zeta) = P(i\tau, \zeta) = 0 \implies (\text{Im } \tau, \text{Im } \zeta) \notin -CN - \Gamma(P, N),$$

Indeed, if $(\tau, \zeta) = \Xi + i(-CN - \theta)$, with $\theta \in \Gamma(P, N)$, $\Xi \in \mathbb{R}^{d+1}$, then $\theta - \epsilon N \in \Gamma(P, N)$ is ϵ is small enough because $\Gamma(P, N)$ is open, and then

$$P(\tau, \zeta) = P(\Xi - i(C + \epsilon)N - i(\theta - \epsilon N)) \neq 0.$$

Hence $-(C + \text{Im } \tau, \text{Im } \zeta) \notin \Gamma(P, N)$, and since the cone $\Gamma(P, N)$ is open, this implies

$$\frac{-(C + \text{Im } \tau, \text{Im } \zeta) \cdot N}{|(C + \text{Im } \tau, \text{Im } \zeta)|} = \frac{-(C + \text{Im } \tau)}{|(C + \text{Im } \tau, \text{Im } \zeta)|} \leq C_1,$$

for some constant $C_1 < 1$, which gives

$$\text{Im } \tau \geq -C - \frac{C_1}{1 - C_1} |\text{Im } \zeta|.$$

Example 5.5. Consider the wave operator $P = \partial_t^2 - \Delta$ in $\mathbb{R} \times \mathbb{R}^d$, hence $\sigma(t, \xi) = \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$. Using some further information on the fundamental solution we can give a very precise result on the matrix decay in dimension $d \leq 3$. Namely, for the window $g(x) = 2^{d/4}e^{-\pi|x|^2}$ ($g \in S_{1/2}^{1/2}(\mathbb{R}^d)$, $\|g\|_{L^2} = 1$), we claim that

$$|\langle \sigma(t, D_x) M_\xi T_x g, M_{\xi'} T_{x'} g \rangle| \leq t e^{-\frac{\pi}{2} [|\xi' - \xi|^2 + (|x' - x| - t)_+^2]}, \quad x, x', \xi, \xi' \in \mathbb{R}^d, \quad d \leq 3,$$

where $(\cdot)_+$ denote positive part. This agrees with Proposition 5.4, with the constants made explicit.

In fact, in dimension $d \leq 3$ we know that

$$\sigma(t, D_x) f(x) = \int f(x - y) d\mu_t(y)$$

where μ_t is a positive Borel measure, supported in the ball $B(0, t)$, with total mass $= t$ (see e.g. [31, Chapter 4]).

Hence, using $T_y M_\xi = e^{-2\pi i y \cdot \xi} M_\xi T_y$,

$$\langle \sigma(t, D_x) M_\xi T_x g, M_{\xi'} T_{x'} g \rangle = \int e^{-2\pi i y \cdot \xi} \langle M_\xi T_{x+y} g, M_{\xi'} T_{x'} g \rangle d\mu_t(y).$$

An explicit computation shows that

$$|\langle \pi(z) g, \pi(w) g \rangle| = e^{-\frac{\pi}{2} |w-z|^2}, \quad z, w \in \mathbb{R}^{2d},$$

so that

$$\begin{aligned} |\langle \sigma(t, D_x) M_\xi T_x g, M_{\xi'} T_{x'} g \rangle| &\leq \int e^{-\frac{\pi}{2} [|\xi' - \xi|^2 + |x' - x - y|^2]} d\mu_t(y) \\ &\leq t e^{-\frac{\pi}{2} [|\xi' - \xi|^2 + (|x' - x| - t)_+^2]}, \end{aligned}$$

which gives the claim.

Consider now the Gabor frame $\mathcal{G}(g, \Lambda)$, with $g(x) = 2^{d/4}e^{-\pi|x|^2}$, $\Lambda = \mathbb{Z}^d \times (1/2)\mathbb{Z}^d$ ([21, Theorem 7.5.3]), and the Gabor matrix

$$T_{m',n',m,n} = \langle \sigma(t, D_x) M_n T_m g, M_{n'} T_{m'} g \rangle, \quad (m, n), (m', n') \in \Lambda.$$

We therefore have

$$|T_{m',n',m,n}| \leq \tilde{T}_{m',n',m,n} := t e^{-\frac{\pi}{2} [n' - n]^2 + (|m' - m| - t)_+^2}, \quad (m, n), (m', n') \in \Lambda, \quad d \leq 3.$$

Figure 1 shows the magnitude of the entries, rearranged in decreasing order, of a generic column, e.g. $\tilde{T}_{m',n',0,0}$ (obtained for $m = n = 0$), at time $t = 0.75$, in dimension $d = 2$. In fact, the same figure applies to all columns, for $\tilde{T}_{m',n',m,n} = \tilde{T}_{m'-m,n'-n,0,0}$. This figure should be compared with [6, Figure 15], where a similar investigation was carried out for the curvelet matrix of the wave propagator on the unit square ($d = 2$) with periodic boundary conditions. It turns out that the Gabor decay is even better, in spite of the fact that we consider here the wave operator in the whole \mathbb{R}^2 .

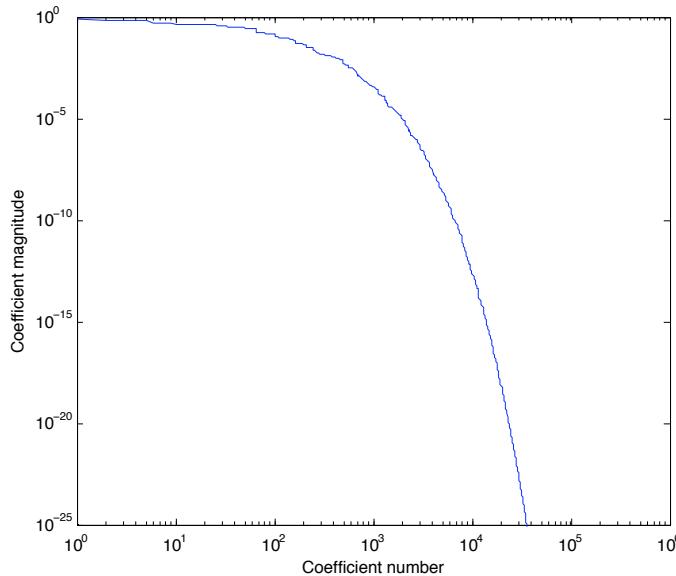


FIGURE 1. Decay of a generic column of the Gabor matrix for $\frac{\sin(2\pi|D|t)}{2\pi|D|}$ in dimension $d = 2$ and at time $t = 0.75$, with window $g(x) = \sqrt{2}e^{-\pi|x|^2}$ and lattice $\mathbb{Z}^2 \times (1/2)\mathbb{Z}^2$.

We now present a class of examples of operators which satisfy Theorem 5.3 and are not hyperbolic.

Example 5.6. Consider the operator

$$(66) \quad P(\partial_t, D_x) = \partial_t + (-\Delta)^k,$$

with $k \geq 1$ integer. When $k = 1$ we have the heat operator. Its symbol is the polynomial

$$P(i\tau, \zeta) = i\tau + (4\pi^2\zeta^2)^k,$$

We claim that it satisfies (59) with $r = 2k$.

If $\tau \in \mathbb{C}$, $\xi, \eta \in \mathbb{R}^d$ and $P(i\tau, \xi + i\eta) = 0$ then

$$\text{Im } \tau = (2\pi)^{2k} \text{Re}(|\xi|^2 + 2i\xi \cdot \eta - |\eta|^2)^k = (2\pi)^{2k}|\xi|^{2k} + Q(\xi, \eta),$$

where $Q(\xi, \eta)$ is a homogeneous polynomial of degree $2k$, with $Q(\xi, 0) = 0$. Hence, for every $\epsilon > 0$ and some constant $C_\epsilon > 0$,

$$|Q(\xi, \eta)| \lesssim \sum_{j+l=2k, j \geq 0, l \geq 1} |\xi|^j |\eta|^l \leq \epsilon |\xi|^{2k} + C_\epsilon |\eta|^{2k},$$

where we applied the inequality $ab \leq \frac{(\epsilon a)^p}{p} + \frac{(b/\epsilon)^q}{q}$ (with $a = |\xi|^j$, $b = |\eta|^l$, $p = 2k/j$ and $q = 2k/l$) to the terms of the sum with $j \geq 1$. Taking $\epsilon = (2\pi)^{2k}/2$ we get

$$\operatorname{Im} \tau \geq \frac{(2\pi)^{2k}}{2} |\xi|^{2k} - C |\eta|^{2k} \geq -C |\eta|^{2k},$$

which proves the claim.

Hence, for the operator P in (66), Theorem 5.3 applies with $r = 2k$ and $s = 1 - 1/(2k)$.

Remark 5.7. Theorems 4.2 and 4.4 consent also some applications to equations with variable coefficients. As elementary example consider the transport equation

$$\begin{cases} \partial_t u - i \left(\sum_{j=1}^d a_j D_{x_j} + \sum_{j=1}^d b_j x_j \right) u = 0 \\ u(0, x) = u_0(x), \end{cases}$$

with $a_j, b_j \in \mathbb{R}$. The fundamental solution of the problem has symbol

$$\sigma(t, x, \xi) = \exp \left[it \sum_{j=1}^d (a_j \xi_j + b_j x_j) \right]$$

satisfying (44) with $m = 1$, $s = 0$, and the preceding arguments apply. More generally, it seems possible to treat similarly the pseudodifferential problems of the form

$$\partial_t - ia^w(x, D)u = 0, \quad u(0, x) = u_0(x),$$

where $a(x, \xi)$ is real-valued and $d_{x, \xi} a(x, \xi)$, as well as higher order derivatives, are bounded in \mathbb{R}^{2d} , satisfying suitable estimates.

□

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